Polynomials 3

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Q> If $P(x)$ denotes polynomial of degree n such that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, \dots, n$ determine

$P(n+1)$.

Ans:— $P(k) = \frac{k}{k+1} \Rightarrow (k+1)P(k) - k = 0$

Let $Q(x) = (x+1)P(x) - x$

$Q(x)$ has zeros at $x = 0, 1, 2, \dots, n$

$\Rightarrow Q(x) = c\,x(x-1)(x-2)\cdots(x-n)$

$\hookrightarrow$ c is a constant

$$\Rightarrow (x+1)P(x) - x = c\,x(x-1)\cdots(x-n) \quad — (1)$$

Putting $x = -1$ we get,

$$(-1+1)P(x) + 1 = c(-1)(-2)\cdots(-n-1) = c(-1)^{n+1}(n+1)!$$

$$\Rightarrow c = \frac{(-1)^{n+1}}{(n+1)!}$$

Putting $x = n+1$ in (1):—

$$(n+2)P(n+1) - (n+1) = \frac{(-1)^{n+1}}{(n+1)!}(n+1)(n)(n-1)\cdots(1)$$

$$\Rightarrow P(n+1) = \frac{\left((-1)^{n+1} + n+1\right)}{n+2}$$

Q) Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system of inequalities

$$(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) \leq 0$$
$$(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) \leq 0$$
$$(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) \leq 0$$
$$(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) \leq 0$$
$$(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) \leq 0$$

Ans:— $(x_i^2 - x_{i+2}x_{i+4})(x_{i+1}^2 - x_{i+2}x_{i+4})$ where indices are read modulo 5.

From the left hand sides we will get $5 \times 4 = 20$ terms of which $\underbrace{10 \text{ perfect}}$ square and $\underbrace{10 \text{ cross-terms}}$.

$$\Downarrow$$

$x_i^2 x_{i+1}^2$ & $x_{i+2}^2 x_{i+4}^2$ $\qquad -x_i^2 x_{i+1} x_{i+3}$ & $-x_i^2 x_{i+2} x_{i+4}$

$$(x_1x_2 - x_1x_4)^2 = x_1^2x_2^2 - 2x_1^2x_2x_4 + x_1^2x_4^2$$

We will write the sum of left hand sides in the form

$$\frac{1}{2}(y_1^2 + y_2^2 + \cdots + y_{10}^2)$$

where each y_k produces a different cross-term and all perfect square terms are exactly duplicated

Finally we get, $$0 \geq \sum_{i=1}^{5} (x_i^2 - x_{i+2}x_{i+4})(x_{i+1}^2 - x_{i+2}x_{i+4})$$
$$= \frac{1}{2}\sum_{i=1}^{5} \left((x_ix_{i+1} - x_ix_{i+3})^2 + (x_{i-1}x_{i+1} - x_{i-1}x_{i+3})^2 \right)$$

We have $0 \geq$ sum of squares $\Rightarrow$ all the squares are 0

$\Rightarrow x_ix_{i+1} = x_ix_{i+3} \Rightarrow x_{i+1} = x_{i+3}$

$\Rightarrow x_{i-1}x_{i+1} = x_{i-1}x_{i+3} \Rightarrow x_{i+1} = x_{i+3}$

$\Rightarrow x_2 = x_4 \quad (i=1)$

$\Rightarrow x_3 = x_5 \quad (i=2)$ $\searrow$

$x_1 = x_2 = x_4 = x_5$ is the solution

$\Rightarrow \quad x_2 = x_4 \quad (i=1)$

$\Rightarrow \quad x_3 = x_5 \quad (i=2)$

$\Rightarrow \quad x_4 = x_1 \quad (i=3)$

$\Rightarrow \quad x_5 = x_2 \quad (i=4)$

$\Rightarrow \quad x_1 = x_3 \quad (i=5)$

$\Rightarrow x_1 = x_2 = x_3 = x_4 = x_5$ is the solution

So the solutions of $(x_1, x_2, x_3, x_4, x_5)$ are,

$\{ (u, u, u, u, u) \mid u \in \mathbb{R} \}$